A Note on the Existence of Absolutely Simple Jacobians

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The purpose of this note is to answer a question to us by M. Saïdi:

Question. Let $g \geq 2$ be a positive integer. Does there exist a projective nonsingular curve of genus g over a finite field whose Jacobian is absolutely simple? If so, are there infinitely many such curves?

There is similar but simpler question, with "finite fields" replaced by "number fields". The answer for both questions is "Yes", see Remark 5 (v).

It is easy to show that there exist infinitely many curves over $\mathbb{C}$ whose Jacobian is simple: the locus in the moduli space $\mathcal{M}_g$ of curves of genus g, consisting of those curves whose Jacobians are not simple, is the union of a countable family of algebraic subvarieties of $\mathcal{M}_g$ of lower dimension. So the union is not equal to $\mathcal{M}_g$.

Of course this naive argument does not work for the countable fields $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}_p}$.

In this note we answer this question affirmatively. We show that for any family of curves $C \to S$ over a base scheme S of finite type over $\mathbb{F}_p$ (or over $\mathbb{Q}$) such that the monodromy is "maximal", there exist infinitely many closed points $s \in S$ such that the Jacobian of C_s is absolutely simple. The proof uses results on ℓ-adic representations, especially about Frobenius tori, due to Serre, Chi, Larsen and Pink. The proof also shows that the assumption on the monodromy can be weakened; this is discussed in Remark 5 (i). Also, the existence of closed points $s \in \mathcal{M}_g(\overline{\mathbb{F}_p})$ with $\mathrm{Jac}(C_s)$ absolutely simple follows from a Chebotarev density argument, so the set of all such points s has positive Dirichlet density in the set $|\mathcal{M}_g|$ of all closed points of $\mathcal{M}_{g,\mathbb{F}_p}$.

Here is a sketch of the basic argument. We are given a family of abelian varieties $A \to S$ in characteristic p with an assumption on its monodromy. The goal is to show that there are many closed fibres in the family which are absolutely simple. If for a closed point $s \in S$, the $\mathbb{Q}_\ell$-Tate module $\mathrm{T}_\ell(A_{\bar{s}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of the fibre A_s is $\mathbb{Q}_\ell$-irreducible under the action of Fr_s^n for each power of the Frobenius Fr_s, then A_s is absolutely simple. Let $G^\natural$ be the set of all semisimple conjugacy classes of the ℓ-adic monodromy group G for $A \to A$. Let $Ir^\natural \subset G^\natural$ be the set of semisimple conjugacy classes $[\gamma]$ such that $\mathrm{T}_\ell(A_{\bar{s}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is irreducible as a $\mathbb{Q}_\ell[\gamma]$ module. If $Ir^\natural$ contains an open subset of $G^\natural$, then we can apply the Chebotarev density theorem to produce fibres A_s in the family which are absolutely simple. In general, whether the subset $Ir^\natural$ is empty is a property of the representation of G on the Tate module. In the exposition below, we assume for simplicity that the monodromy representation is the standard representation of the group of symplectic similitudes. In this case $Ir^\natural$ contains the conjugacy classes of all generators of elliptic maximal tori, and Krasner's lemma implies the openness we need.

Notations: Let S be a geometrically connected scheme smooth over a finite field $\mathbb{F}_q \supseteq \mathbb{F}_p$. Let $(\pi : A \to S, \lambda : A \to A^t)$ be a polarized abelian scheme over S of relative dimension g. Let η (resp. $\bar{\eta}$) be the generic point of S (resp. a geometric generic point of S). Let ℓ be a prime number different from p. Let $\langle , \rangle_\lambda$ be the symplectic pairing on the Tate module $\mathrm{T}_\ell(A_{\bar{\eta}}) \otimes \mathbb{Q}_\ell$ attached to the polarization λ. Let $*_{A,\lambda}$ be the symplectic involution on the endomorphism algebra $\mathrm{End}(\mathrm{T}_\ell(A_{\bar{\eta}}) \otimes \mathbb{Q}_\ell)$ defined by $\langle , \rangle_\lambda$. Let $\mathrm{Sp}_{A,\lambda}$ (resp. $\mathrm{GSp}_{A,\lambda}$) be the symplectic group (resp. the group of symplectic similitudes) in $2g$ variables defined by $*_{A,\lambda}$. Let $\rho : \pi_1(S, \bar{\eta}) \to \mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$ be the ℓ-adic representations attached to $\mathrm{T}_\ell(A_{\bar{\eta}})$. Recall that the fundamental group $\pi_1(S, \bar{\eta})$ fits canonically into an exact sequence

$$1 \longrightarrow \pi_1(\overline{S}, \bar{\eta}) \longrightarrow \pi_1(S, \bar{\eta}) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \longrightarrow 1 \quad ,$$

where $\overline{S} = S \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}_q}$.

Let $G = G_A = \rho(\pi_1(S))$ (resp. $G_1 = G_{A,1} = \rho(\pi_1(\overline{S}, \bar{\eta}))$) be the image of the fundamental group (resp. the geometric fundamental group) in $\mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$. Let $G^{\mathrm{alg}} = G_A^{\mathrm{alg}}$ (resp. $G_1^{\mathrm{alg}} = G_{A,1}^{\mathrm{alg}}$) be the Zariski closure of G_A (resp. $G_{A,1}$) in $\mathrm{GSp}_{A,\lambda}$.

Construction/Definition: For each closed point $s \in S$, there is a Frobenius element Frob_s in $\pi_1(S, \bar{\eta})$ attached to s, well-defined up to conjugation.

The neutral component of the $\mathbb{Q}_\ell$-Zariski closure of the subgroup of $\mathrm{GSp}_{A,\lambda}$ generated by a Frobenius element $\rho(\mathrm{Frob}_s)$ is a subtorus of $\mathrm{GSp}_{A,\lambda}$; therefore it is also a subtorus of the Zariski closure G_A^{alg}. This subtorus T_s of G_A^{alg} is well-defined up to conjugation in $G_A^{\mathrm{alg}}(\mathbb{Q}_\ell)$. We call it "the" *Frobenius torus* attached to the closed point s, and denote it by T_s.

Remark 1. The notion of Frobenius torus is due to Serre, see [4, §3], [7, §4, §7]. The Frobenius torus T_s can be defined over $\mathbb{Q}$; here we use its "ℓ-adic realization".

In the next lemma we show that under suitable conditions on the monodromy of $A \to S$, there are many Frobenius tori T_s which are elliptic maximal tori; in other words T_s is anisotropic modulo the center of $\mathrm{GSp}_{2g,\lambda}$. We denote by $|S|$ the set of all closed points of S, and by $|S|_{\mathrm{emFt}}$ the subset of all $s \in |S|$ such that the Frobenius torus T_s is an elliptic maximal torus of $\mathrm{GSp}_{A,\lambda}$ over $\mathbb{Q}_\ell$.

Lemma 2. *Notations as above. Assume that the Zariski closure of the image $\rho(\pi_1(\overline{S}, \bar\eta))$ of the geometric fundamental group is equal to $\mathrm{Sp}_{A,\lambda}$. Then $|S|_{emFt}$ has positive Dirichlet density in $|S|$. In particular $|S|_{emFt}$ is an infinite set.*

Proof. This statement is certainly known. We reproduce a proof for the convenience of the reader. As a first step, the assumption implies that G is an open subgroup in $G^{\mathrm{alg}}(\mathbb{Q}_\ell) = \mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$. Admit this for the moment.

The $\mathbb{Q}_\ell$-rational maximal tori in $\mathrm{GSp}_{A,\lambda}$ are given by maximal commutative semisimple subalgebras B in $\mathrm{End}(\mathrm{T}_\ell(A_{\bar\eta}) \otimes \mathbb{Q}_\ell)$ which are stable under the involution $*_{A,\lambda}$. For any such subalgebra B, the $\mathbb{Q}_\ell$-subspace of elements fixed under $*_{A,\lambda}$ has dimension $g = \frac{1}{2}\dim_{\mathbb{Q}_\ell}(B)$. Among these maximal tori, the elliptic ones correspond to subfields of dimension $2g$ over $\mathbb{Q}_\ell$, and they are known to exist. For any element $x \in \mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$, the neutral component of the Zariski closure of the subgroup generated by x is an elliptic maximal torus if and only if the algebra $\mathbb{Q}_\ell[x] \subset \mathrm{End}(\mathrm{T}_\ell(A_{\bar\eta}) \otimes \mathbb{Q}_\ell)$ is a field. Let Em be the subset of G consisting of all elements of $x \in G$ such that $\mathbb{Q}_\ell[x] \subset \mathrm{End}(\mathrm{T}_\ell(A_{\bar\eta}) \otimes \mathbb{Q}_\ell)$ is a field of degree $2g$ over $\mathbb{Q}_\ell$. By Krasner's lemma [6, Prop. 3, p. 43], Em is an open subset of G. More concretely, if $\mathbb{Q}_\ell[x_0]$ is a field of degree $2g$ over $\mathbb{Q}_\ell$, then for all $x \in \mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$ sufficiently close to x_0 in the ℓ-adic topology, the characteristic polynomial and the eigenvalues of x are close to those for x_0. So $\mathbb{Q}_\ell[x]$ is a field of degree

at least $2g$, therefore the degree is exactly $2g$. It is known that Em is not empty; so there exists an open normal subgroup $N \subset G$ such that Em is a union of cosets for N.

Recall that Chebotarev's density theorem states that for any finite étale Galois covering $X \to Y$ between normal integral schemes of finite type over $\mathbb{Z}$, the Frobenius conjugacy classes in the covering group attached to the closed points of Y is equidistributed, see [9, Thm. 7, p. 91]. Applying this density theorem to the finite étale covering of S corresponding to the finite quotient $G \to G/N$ of the monodromy group G, we see that there exist infinitely many Frobenius tori which are elliptic maximal tori in $\mathrm{GSp}_{A,\lambda}$ over $\mathbb{Q}_\ell$. This finishes the proof of Lemma 2, except for the fact that G is open in $G^{\mathrm{alg}}(\mathbb{Q}_\ell)$. This is done in the next two paragraphs. The reader may wish to skip them because it is a little technical and not central to this note; cf. Remark 3 (i).

Both G and G_1 are closed subgroups of $\mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$; hence they are analytic Lie subgroups of $\mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$. Let $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) be the Lie algebra of G (resp. G_1), and let $\mathfrak{g}^{\mathrm{alg}}$ (resp. $\mathfrak{g}_1^{\mathrm{alg}}$) be the Lie algebra of G^{alg} (resp. G_1^{alg}). The assumption of the theorem means that $\mathfrak{g}_1^{\mathrm{alg}}$ is equal to the Lie algebra $\mathfrak{sp}_{2g,\lambda}$ of $\mathrm{Sp}_{A,\lambda}$. Therefore $\mathfrak{g}^{\mathrm{alg}}$ is equal to the Lie algebra $\mathfrak{gsp}_{2g,\lambda}$ of $\mathrm{GSp}_{A,\lambda}$.

By [3, §7, Cor. 7.9], we have $[\mathfrak{g}_1^{\mathrm{alg}}, \mathfrak{g}_1^{\mathrm{alg}}] \subseteq \mathfrak{g}_1$. We reproduce a proof here. The algebraic subgroup of $\mathrm{GSp}_{A,\lambda}$ consisting of all elements $x \in \mathrm{GSp}_{A,\lambda}$ such that $\mathrm{Ad}(x)(\mathfrak{g}_1) \subseteq \mathfrak{g}_1$ clearly contains G, hence $[\mathfrak{g}_1^{\mathrm{alg}}, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$. Repeating this argument again, one concludes that $[\mathfrak{g}_1^{\mathrm{alg}}, \mathfrak{g}_1^{\mathrm{alg}}] \subseteq \mathfrak{g}_1$.

From $[\mathfrak{g}_1^{\mathrm{alg}}, \mathfrak{g}_1^{\mathrm{alg}}] \subseteq \mathfrak{g}_1$, we deduce that $\mathfrak{g}_1 = \mathfrak{sp}_{2g,\lambda}$ and $\mathfrak{g} = \mathfrak{gsp}_{2g,\lambda}$. Especially G is an open subgroup of $\mathrm{GSp}_{A,\lambda}(\mathbb{Q}_\ell)$. $\square$

Remarks 3. (i) The argument to show that $\mathfrak{g}_1$ is big since it contains the derived algebra of its algebraic envelope appeared in [2].

(ii) It is easy to find examples which satisfy the assumption in Lemma 2. For instance, one can take S to be equal to the moduli space $\mathcal{M}_{g,n}$ of curves of genus g over $\mathbb{F}_p$ with the principal symplectic level-n structure, or a generic curve in $\mathcal{M}_g$, with $n \geq 3$ and $(n, p\ell) = 1$. In this case the monodromy group G is equal to $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$; this is equivalent to [5, Thm. 5.15, p. 108], that the moduli spaces $\mathcal{M}_{g,n\ell^i}$ are geometrically connected. Then Proposition 4 below answers the Question at the beginning of this note.

Proposition 4. *Notation and assumptions as in Lemma 2. Then for each $s \in |S|_{emFt}$, the fiber A_s is an absolutely simple abelian variety.*

Proof. Suppose that $s \in S$ is a closed point such that the Frobenius torus T_s is an elliptic maximal torus in $\mathrm{GSp}_{A,\lambda}$ over $\mathbb{Q}_\ell$. We know that each elliptic maximal torus T in $\mathrm{GSp}_{A,\lambda}$ is given by a subfield F in $\mathrm{End}(\mathrm{T}_\ell(A_{\bar{\eta}}) \otimes \mathbb{Q}_\ell)$ of degree $2g$ stable under $*_{A,\lambda}$, hence T acts irreducibly on $\mathrm{T}_\ell(A_{\bar{\eta}}) \otimes \mathbb{Q}_\ell$. So the action of any open subgroup of $\mathrm{Gal}(\bar{s}/s)$ on the ℓ-adic Tate module of A_s is irreducible over $\mathbb{Q}_\ell$. This implies that A_s is not isogenous to a non-trivial product over any finite extension of the residue field $\kappa(s)$ of s. $\square$

Remarks 5. (i) In the proof above, what we really need about the monodromy of $\pi : A \to S$ is that the neutral component G^{alg} of the Zariski closure of $\rho(\pi_1(S))$ has an elliptic maximal torus T over $\mathbb{Q}_\ell$ such that the restriction to T of the representation of G^{alg} on the ℓ-adic Tate module is irreducible. For instance, if G^{alg} is isomorphic to the derived group $(\mathrm{Res}_{E/\mathbb{Q}_\ell}\mathrm{GL}_2)^{\mathrm{der}}$ of $\mathrm{Res}_{E/\mathbb{Q}_\ell}\mathrm{GL}_2)$ for a finite extension E of $\mathbb{Q}_\ell$ and the representation is isomorphic to the standard representation of $\mathrm{Res}_{E/\mathbb{Q}_\ell}\mathrm{GL}_2)$, the argument still works and one gets absolutely simple fibers in the abelian scheme in question.

(ii) On the other hand, suppose $\pi : A \to S$ is the universal family of abelian surfaces over the Shimura curve S attached to a quaternion division algebra B over $\mathbb{Q}$, such that B is split at p, and ℓ is a prime number different from p. Then the representation of every Frobenius torus T_s on the ℓ-adic Tate module is reducible. The group G^{alg} is the group attached to $B^{\mathrm{opp},\times}$ where B^{opp} is the opposite algebra of B, and the representation on the ℓ-adic Tate module comes from the regular representation of B^{opp}. In this example one can show that none of the fibres A_s is absolutely simple.

(iii) If the representation of the Frobenius torus T_s on the ℓ-adic Tate module is reducible, then $\mathrm{End}(A_{\bar{s}}) \otimes \mathbb{Q}_\ell$ contains a nontrivial idempotent. But this in general does not imply that $\mathrm{End}(A_{\bar{s}}) \otimes \mathbb{Q}$ contains a nontrivial idempotent, so A_s over the closed point s may or may not be absolutely simple.

(iv) If A_K is an abelian variety of dimension g over a number field such that the Zariski closure of the image of the ℓ-adic Galois representation attached to A is equal to GSp_{2g}, the same argument shows that there exist finite places v of K such that the reduction A_v of A at v is an absolutely simple abelian variety; such places v form a subset of positive density. One can also change GSp_{2g} to a group satisfying the property specified in (i).

By [8, 1.4], due to Serre, for any abelian variety A_L over a field L of finite type over $\mathbb{Q}$, there exist specializations B_K to number fields K, such that the images of the ℓ-adic Galois representations for A_L and B_K are equal. Therefore there exist examples satisfying the assumption in the previous paragraph.

(v) Proposition 4 gives an affirmative answer to the Question at the beginning of this note. The similar question, with $\overline{\mathbb{F}_p}$ replaced by $\overline{\mathbb{Q}}$, was answered by [8, Cor. 1.5]. and also by [1, Thm. 0.6.1, p. 8]. One can also deduce it from the $\overline{\mathbb{F}_p}$-version because of Lemma 6 below.

Lemma 6. *Suppose that A_K is an abelian variety over a number field K with good reduction at a finite place v of K. If the reduction A_v of A is absolutely simple, then the abelian variety A is also absolutely simple.*

Proof. Since the hypothesis does not change if one extends the base field K to a bigger number field L, it suffices to prove that the abelian variety A is simple over K. Recall that whether the abelian variety A is simple or not is a question on the existence of an endomorphism T of A such that $T^2 = n \cdot T$ for some non-zero integer n, while $\mathrm{Ker}(T)$ is not finite. Since every endomorphism of A over K extends to an endomorphism of the abelian scheme A over $\mathcal{O}_{K,v}$ which extends A, we are done. $\square$

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